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ORIGINAL PAPER

Counting oriented rectangles and the propagation of waves

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Abstract We propose a simple counting problem involving chains of rectangles on a planar lattice. The boundaries of the chains form a type of random walk with a finite inner scale. With orientation neglected, the continuum limit of the walk densities obeys the Telegraph equation, a form of diffusion equation with a finite signal velocity. Taking into account the orientation of the rectangles, the same continuum limit yields the Dirac equation. This provides an interesting context in which the Dirac equation is phenomenological rather than fundamental.

Keywords Wave-particle duality · Dirac equation

1 Introduction

As one of Stu Whittington's graduate student in the early '80's, I was extremely fortunate to have a supervisor who had a deep intellectual interest in science in general. As his student, I constantly discussed work with Stu that was a long way from both my thesis topic and from Stu's primary areas of interest. He never complained about this, indeed he always tackled each new problem or idea with interest, insight and humour.

The work I want to discuss today is an offshoot of one of those discussions, and it relates to one of Stu's quotes from that era.

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Table 1 The relation between Classical PDE's based on stochastic models, and their 'Quantum' cousins. Formal analytic continuation transforms one to the other, but then the stochastic basis for the equations becomes formal

'Ontology'	Classical	Quantum
	Kac (Poisson)	?
First order	$\frac{\partial \mathbf{U}}{\partial t} = c\,\boldsymbol{\sigma}_z \frac{\partial U}{\partial z} + a\,\boldsymbol{\sigma}_x U$	$\frac{\partial \Psi}{\partial t} = c\,\boldsymbol{\sigma}_z \frac{\partial \Psi}{\partial z} + i\,m\,\boldsymbol{\sigma}_x \Psi$
Second order	$\frac{\partial^2 U}{\partial t^2} = c^2 \frac{\partial^2 U}{\partial z^2} + a^2 U$	$\frac{\partial^2 \psi}{\partial t^2} = c^2 \frac{\partial^2 \psi}{\partial z^2} + (i\,m)^2 \psi$
'Non-relativistic'	$\frac{\partial U}{\partial t} = D \frac{\partial^2 U}{\partial x^2}$	$\frac{\partial \psi}{\partial t} = i\,D \frac{\partial^2 \psi}{\partial x^2}$

Note that the Quantum equations are regarded as *fundamental*, the Classical equations are phenomenologies

"Statistical Mechanics is *easy*. All you ever have to do is count." ... "The trick of course is to find the right objects to count!" SGW 1980

Applying this statement to quantum mechanics gives an interesting perspective on the difference between classical and quantum physics. In Table 1, the equations in the 'classical' column are partial differential equations describing aspects of diffusion. The first two equations are forms of the telegraph equations that describe diffusive processes with a fixed finite mean free speed c. The 'non-relativistic' limit describes the same process in the limit that c is far larger than any speed observable in the system. The resulting PDE is the diffusion equation. One unifying feature of all of these PDE's is that their solutions may be understood by classical statistical mechanics. All solutions may be found by counting paths on an underlying lattice and taking a suitable continuum limit. The solutions are easy to understand, all you have to do is count paths. In a sense, the PDE's may be regarded as phenomenological descriptions of the underlying stochastic processes (random walks) that one uses to derive the equations.

In comparison, the equations in the right-hand column, though formally similar, are not so easy to understand. They are fundamental equations (as opposed to phenomenologies) in that in their quantum contexts, the solutions (wavefunctions) are considered to contain *all* the information about the quantum system ...there is no stochastic process 'underneath' the quantum equations. The closest we come to such an underlying process in quantum mechanics is the Feynman sum-over-paths formulation that strongly resembles the Wiener integral for the diffusion equation. The analogy falls short of providing a real underlying process however since paths are not counted by natural numbers, they are used to propagate phase in spacetime. The presence of the unit imaginary i in the PDE's in the right hand column in Table 1 signals the implication of complex solutions, outside the domain of the probability density function solutions expected for the classical equations.

The quantum equations in the right-hand column are wave equations, and it might seem that if we insist on trying to derive the equations as limiting cases of counting objects, the objects themselves should be waves. However the formal similarity between the classical and quantum equations is suggestive of a particle picture, as is the measurement process that must eventually be invoked. Thus we shall sketch the derivation of the classical equation in the first row of Table 1 by counting paths using a model due to Marc Kac. We shall then show that a relatively minor, but somewhat surprising change in what is being counted, allows us to find a classical context for the Dirac equation.

2 The telegraph equations

A kinetic theory basis for the telegraph equations was first explored by Kac [1]. Consider a random walk in one dimension where the particle hops from lattice site to lattice site at some fixed speed c. The probability of a direction change is proportional to the lattice spacing ϵ, ie. $p = m\,\epsilon$. The number of direction changes per unit time is then Poisson distributed for small ϵ and the paths themselves appear as sketched in Fig. 1.

We shall keep track of two probability densities corresponding to the two directions of motion.

$$U = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}$$

On the lattice the walk is Markovian with transition matrix:

$$T = \begin{pmatrix} 1 - m\,\epsilon & m\,\epsilon \\ m\,\epsilon & 1 - m\,\epsilon \end{pmatrix}. \tag{1}$$

To count walks we form the generating function (Fourier Transform), to lowest order in ϵ, this gives us the transfer matrix:

$$T_k = (1 - m\,\epsilon)(\mathbf{1} - \epsilon(k\sigma_z - m\,\sigma_x)) \tag{2}$$

where the σ_k are the usual Pauli matrices. In the continuum limit at fixed t:

$$T_k^{t/\epsilon} \to e^{-m\,t}\left(\mathbf{1}\,\cosh(Et) - \frac{k}{E}\sinh(Et)\sigma_z + \frac{m}{E}\sinh(Et)\sigma_x\right) \tag{3}$$

with $E = \sqrt{m^2 - k^2}$ and $\mathbf{1}$ the unit matrix. This ultimately gives the Telegraph equations . . . diffusion with a finite signal velocity and mean free path.

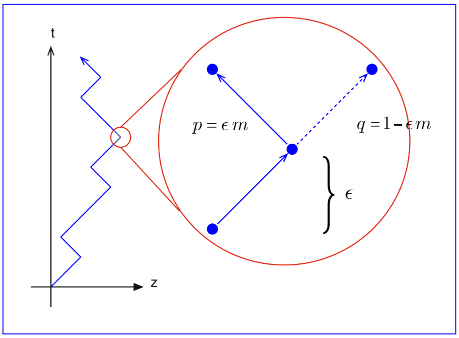


Fig. 1 Here is a sketch of one of the Kac walks. Ensemble averages lead to the Telegraph equations: $\frac{\partial \mathbf{U}}{\partial t} = c\,\boldsymbol{\sigma}_z \frac{\partial U}{\partial z} + m\,\boldsymbol{\sigma}_x U$ which express conservation of particle number

3 The Dirac equation

The relation between the telegraph equations and the Dirac equation through formal analytic continuation was first examined by Gaveau et al.[2]. Extensions that bypass analytic continuation may be found in [3–5]. Proceeding along the same lines as the latter, we have just seen that the two-component Telegraph equations (The first equations in Table 1) result from counting the walks illustrated in Fig. 1, and we ask the question, 'is there anything we can count that will give us the Dirac equation instead of the telegraph equations?' The answer is yes! Consider Fig. 2. Here we form a chain of oriented rectangles using the same Poisson process that we used in the Kac walks. Notice that the merged boundaries of the rectangles form a continuous curve that may be traversed from $t = 0$ to some large value of t and back again, forming the chain of oriented rectangles. Furthermore, the process may be repeated to cover the future light-cone above the origin with an ensemble of oriented rectangles. Counting using an orientation variable that is ± 1 allows us to calculate a net orientation. That is, imagine standing at some point in the (z, t) plane in the future cone from the origin after the stochastic process has generated many oriented rectangles near your position. You will find that you are inside some rectangles with positive orientation, and inside some with negative orientation. If you count the number of positively oriented rectangles, subtract the number of negatively oriented rectangles and divide by the total number, you will get a net orientation that is between ± 1. This value will change depending on where you are in the (z, t) plane and you might expect that small changes in position would give small changes in the net orientation resulting in a smooth, wave-like change in the net orientation.

To see how this comes about, recall for the Kac 2-state walk we started with the simple transition matrix of Eq. 1. From Fig. 2 we see that when we count oriented

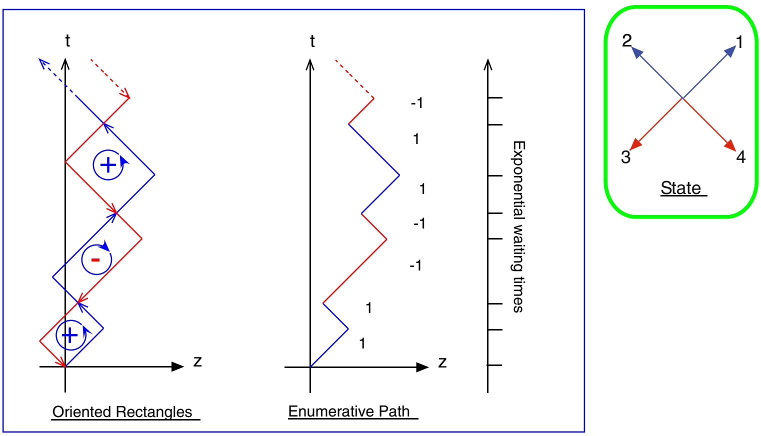
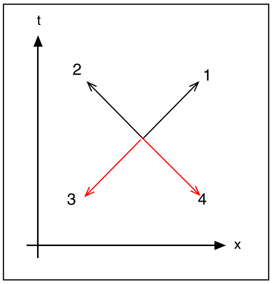


Fig. 2 We can form a chain of oriented rectangles using the same stochastic process as above. The alternating orientation is counted as either +1 or −1. The oriented boundaries form a continuous path from the origin out to some large value of t and back again. If the cycle is repeated many times the (z, t) plane inside the 'future light cone' will be covered by an ensemble of oriented rectangles. Counting orientation as opposed to simple path density gives the Dirac equation



Transition Matrix for four states.

$$T = \begin{pmatrix} 1 - m\epsilon & m\epsilon & 0 & 0 \\ 0 & 1 - m\epsilon & m\epsilon & 0 \\ 0 & 0 & 1 - m\epsilon & m\epsilon \\ m\epsilon & 0 & 0 & 1 - m\epsilon \end{pmatrix}$$

After change of variables.

$$T_c = \begin{pmatrix} 1 - m\epsilon & m\epsilon & 0 & 0 \\ m\epsilon & 1 - m\epsilon & 0 & 0 \\ 0 & 0 & 1 - m\epsilon & m\epsilon \\ 0 & 0 & -m\epsilon & 1 - m\epsilon \end{pmatrix}$$

Fig. 3 The Markov chain now has four states and the walk cycles through the four spacetime directions. After a change of variables to sums and differences by direction the cyclic nature of the walk is manifest in the lower block! The lower block is $(1 - m\,\epsilon)(\mathbf{1} + m\,\epsilon\,\sigma_z\,\sigma_x)$

squares via the 'Enumerative path' formed by the concatenation of the right-hand boundaries of the rectangles, there are then four states corresponding to the four space-time directions. The new transition matrix is shown in Fig. 3. If one changes variables from the occupation densities $u_1, \ldots, u_4$ to sums and differences of these densities, $(u_1 + u_3)$, $(u_2 + u_4)$ and $(u_1 - u_3)$, $(u_2 - u_4)$ The sums are still densities and reproduce the Kac transition matrix (1) with the resulting continuum limit (3).

For comparison, the lower block 'densities' have the transfer matrix

$$T_F = (1 - m\,\epsilon)(\mathbf{1} - \epsilon(k\sigma_z - m\,\sigma_z\,\sigma_x))$$

In the continuum limit

$$T_F^{t/\epsilon} \to e^{-m\,t}(\mathbf{1}\cos(Et) - \frac{k}{E}\sin(Et)\sigma_z + \frac{m}{E}\sin(Et)\sigma_z\,\sigma_x) \tag{4}$$

with $E = \sqrt{m^2 + k^2}$. Note $(\sigma_z\,\sigma_x)^2 = -\mathbf{1}$ and the product of σ_z and σ_x serves the same function as the unit imaginary in the Dirac equation in Table 1. Note that apart from an exponential decay that is easily removed, the continuum limit in (4) satisfies the Dirac equation.

An interesting feature of this simple model is that ultimately the appearance of both wave propagation and Lorentz covariance are a result of counting the right objects ... oriented rectangles whose geometry is determined by an underlying Poisson process.

Whence QM?

A fundamental assumption of quantum mechanics is that the wavefunction provides *all the available information about a physical system.* Apparently we need no more information than we can get through the wavefunction, and perhaps there simply is no more, even in principle! It is not known whether there is any actual object in the physical world that a wavefunction is mimicking. In the end, wavefunctions in quantum mechanics are just abstract elements in a probability calculus.

Fig. 4 Chains of oriented rectangles can be thought of as chains of virtual pair creation-annihilation events

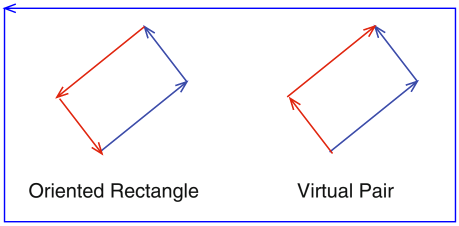


Table 2 Comparison of features appearing in the counting of oriented rectangles and in quantum mechanics

Feature	Oriented rectangles	Quantum mechanics
Wavefunction	Stochastic orientation	Fundamental
Phase	Smoothed direction	Wave-particle duality
Special relativity	Particle speed c fixed	Assumed
Lorentz	Poisson distribution	Assumed
Dirac Sea	Oriented rectangles	Interpretation
Wave-particle duality	Single path, global pattern	Interpretation

In contrast, the Dirac equation appears in this talk as a phenomenology ... an idealization of the average orientation of Stochastic rectangles generated by a single spacetime path. In terms of more conventional pictures of the Dirac equation, the oriented rectangles organize a “Dirac Sea” of virtual particles into an ensemble of boundaries traversed by a single curve (Fig. 4). The intrinsic phase in this model is discrete, corresponding to the four directions in a two dimensional spacetime. The Pauli matrices serve as vectors and the unit imaginary is replaced by the bivector $(\sigma_z \sigma_x)$ whose square is negative. The effect of the ensemble of rectangles is to smooth the discrete phase to produce the analog of complex numbers. and is ultimately a result of “Special Relativity” that is here implicated by choosing a fixed particle speed in conjunction with a geometry determined by the Poisson process.

The Dirac equation in Quantum Mechanics is fundamental ... there is no known statistical process underneath it. In the oriented rectangle derivation, there is an underlying counting problem that allows us to identify the origin of certain features. For example in quantum mechanics the wavefunction is a fundamental object that is a solution of a relevant differential equation. In the Oriented rectangle context it contains densities of oriented rectangles in a continuum limit. In Quantum mechanics phase appears because the fundamental equations are wave equations. In the Oriented rectangle context, phase appears as a statistical averaging over the four spacetime directions of the problem. The four directions give rise to a natural orientation which, in turn, conspires with the Poisson process to extract a smooth wave-like behaviour in long-path averages. Some of these features are compared in Table 2.

4 Conclusions

Statistical mechanics is about counting recognizable objects. The diffusion and telegraph equations are continuum limits of difference equations that count appropriate forms of random walks. As a result, solutions of these differential equations may be understood in terms of the counting of paths.

The Dirac equation in 1 + 1 dimension, although intrinsically fundamental in the context of quantum mechanics, may be derived as the limit of a stochastic process that counts oriented rectangles. The derivation allows us to associate wavefunctions with actual densities that may be constructed stochastically through numerical simulation[6]. This is of interest because it potentially gives us a new perspective on solutions, both approximate and exact, of the quantum wave equations. Being based completely in a particle paradigm, it also holds out the possibility that the 'measurement problem' [7] of quantum mechanics may be more transparent in this new context.

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